Machine Learning for Finance – Problem Set 2

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Instructions. Do not refer to any outside sources to complete this assignment, in accordance with the honor code. If you discussed any problems with other students, indicate that in your solutions.

1. *Hyperplanes.* What is the distance between two parallel hyperplanes $\{x \in \mathbf{R}^n \mid a^T x = b_1\}$ and $\{x \in \mathbf{R}^n \mid a^T x = b_2\}$?

2. *Voronoi description of halfspace.* Let a and b be distinct points in $\mathbf{R}^n$. Show that the set of all points that are closer (in Euclidean norm) to a than b, *i.e.*, $\{x \mid \|x - a\|_2 \leq \|x - b\|_2\}$, is a halfspace. Describe it explicitly as an inequality of the form $c^T x \leq d$.

3. Which of the following sets are convex?

   (a) A *slab*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha \leq a^T x \leq \beta\}$.

   (b) A *rectangle*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha_i \leq x_i \leq \beta_i,\ i = 1, \ldots, n\}$. A rectangle is sometimes called a *hyperrectangle* when $n > 2$.

   (c) A *wedge*, *i.e.*, $\{x \in \mathbf{R}^n \mid a_1^T x \leq b_1,\ a_2^T x \leq b_2\}$.

   (d) The set of points closer to a given point than a given set, *i.e.*,

   $$\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\}$$

   where $S \subseteq \mathbf{R}^n$.

4. *Some sets of probability distributions.* Let x be a real-valued random variable with $\mathrm{P}(x = a_i) = p_i$, $i = 1, \ldots, n$, where $a_1 < a_2 < \cdots < a_n$. Of course $p \in \mathbf{R}^n$ lies in the standard probability simplex $P = \{p \mid \mathbf{1}^T p = 1,\ p \succeq 0\}$. Which of the following conditions are convex in p? (That is, for which of the following conditions is the set of $p \in P$ that satisfy the condition convex?)

   (a) $\alpha \leq \mathrm{E}[f(x)] \leq \beta$, where $\mathrm{E}[f(x)]$ is the expected value of $f(x)$, *i.e.*, $\mathrm{E}[f(x)] = \sum_{i=1}^n p_i f(a_i)$. (The function $f : \mathbf{R} \to \mathbf{R}$ is given.)

   (b) $\mathrm{P}(x > \alpha) \leq \beta$.

   (c) $\mathrm{E}[|x^3|] \leq \alpha \mathrm{E}[|x|]$.

   (d) $\mathrm{E}[x^2] \leq \alpha$.

(e) $\mathrm{E}[x^2] \geq \alpha$.

(f) $\mathbf{var}(x) \leq \alpha$, where $\mathbf{var}(x) = \mathrm{E}[x - \mathrm{E}[x]]^2$ is the variance of x.

(g) $\mathbf{var}(x) \geq \alpha$.

5. *Some functions on the probability simplex.* Let x be a real-valued random variable which takes values in $\{a_1, \dots, a_n\}$ where $a_1 < a_2 < \cdots < a_n$, with $\mathrm{P}(x = a_i) = p_i$, $i = 1, \dots, n$. For each of the following functions of p (on the probability simplex $\{p \in \mathbf{R}^n_+ \mid \mathbf{1}^T p = 1\}$), determine if the function is convex or concave.

   (a) $\mathrm{E}[x]$.

   (b) $\mathrm{P}(x \geq \alpha)$.

   (c) $\mathrm{P}(\alpha \leq x \leq \beta)$.

   (d) $\sum_{i=1}^n p_i \log p_i$, the negative entropy of the distribution.

   (e) $\mathbf{var}\, x = \mathrm{E}[x - \mathrm{E}[x]]^2$.

6. *Some simple LPs.* Give an explicit solution of each of the following LPs.

   (a) *Minimizing a linear function over a halfspace.*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a^T x \leq b, \end{array}$$

   where $a \neq 0$.

   (b) *Minimizing a linear function over a rectangle.*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & l \preceq x \preceq u, \end{array}$$

   where l and u satisfy $l \preceq u$.

   (c) *Minimizing a linear function over the probability simplex.*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{1}^T x = 1, \quad x \succeq 0. \end{array}$$

   What happens if the equality constraint is replaced by an inequality $\mathbf{1}^T x \leq 1$?

   We can interpret this LP as a simple portfolio optimization problem. The vector x represents the allocation of our total budget over different assets, with x_i the fraction invested in asset i. The return of each investment is fixed and given by $-c_i$, so our total return (which we want to maximize) is $-c^T x$. If we replace the budget constraint $\mathbf{1}^T x = 1$ with an inequality $\mathbf{1}^T x \leq 1$, we have the option of not investing a portion of the total budget.